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RESTRICTIONS OF STABLE BUNDLES

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Let E be a stable vector bundle on a projective variety X . The theorem of [MR84] ensures that the restriction of E to a general, sufficiently high degree complete intersection curve in X is again stable. The original proof did not give estimates for the “sufficiently high degree,” but such bounds were developed later. There are many special cases where the unstable restrictions are fully understood, but the following three results give the best known estimates for arbitrary bundles. For simplicity we state them only for surfaces, which seems to be the most subtle case.

1 (General effective results). Let X be a normal, projective surface, over an algebraically closed, $|H|$ a very ample linear system and E a stable vector bundle (or reflexive sheaf) of rank r on X . Then

- (1) [Fle84, Lan10] $E|_{D_m}$ is semistable for general $D_m \in |mH|$ for $m \geq C_1$ where C_1 is roughly $\frac{1}{4}r^2(H^2)$.
- (2) [Bog94, Lan04] $E|_{D_m}$ is stable for every smooth $D_m \in |mH|$ for $m \geq C_2$ where C_2 is roughly $\Delta(E) := 2rc_2(E) - (r-1)c_1^2(E)$ (plus a more complicated correction term in positive characteristic).
- (3) [BP11] Assume in addition that the characteristic is 0. Then $E|_{D_m}$ is stable and has the same holonomy group [BK08] as E for every smooth $D_m \in |mH|$ for $m \geq C_3$ where C_3 is roughly $r^{r^2} \cdot \Delta(E)$.

We refer to the original papers for more precise results and other related bounds; see [HL97, Sec.7] for an introduction.

For many applications, for instance for general boundedness results for sheaves, any effective estimate is useful, but it would be of interest to understand the optimal bounds on m in any of the above settings. The aim of this note is to prove Flenner-type theorems that yield stability and then suggesting a possible optimal result along these directions.

Theorem 2. *Let X be a normal, projective surface over an algebraically closed field and $|H|$ an ample and base point free linear system. Let E be a stable reflexive sheaf of rank r on X such that $E|_C$ is semistable for general $C \in |H|$. Then $E|_{D_m}$ is stable for general $D_m \in |mH|$ for $m \geq \frac{1}{2}r^2 + 4$.*

Theorem 3. *Let X be a normal, projective surface, over an algebraically closed field of characteristic 0 and $|H|$ an ample and base point free linear system. Let E be a stable reflexive sheaf of rank r on X such that $E|_C$ is stable (or polystable) for general $C \in |H|$. Then $E|_{D_m}$ is stable and has the same holonomy group as E for general $D \in |mH|$ for $m \geq 2r^2 + 3$.*

In concrete situations, we think of Theorems 2–3 as building on a Flenner-type estimate. Thus first one establishes the semistability of restrictions and then, using the above results one gets stability and the correct holonomy group for general $D_m \in |mH|$ with a polynomial bound on m .

A common feature of Theorems 2–3 is that they use only *one* numerical invariant, the rank of the bundle in the estimate. We do not know how to eliminate (H^2) from the semistability bound in (1.1).

It seems easy to improve the constants in the *quadratic* bounds of Theorems 2–3 by more attention to details. We did not try to optimize our proof since we believe that our approach should yield a *linear* bound in both cases. Even that, however, may not be optimal. In order to call attention to how little is known, let us pose the following (either bold or foolishly optimistic) question.

Question 4. Let X be a normal, projective surface, $|H|$ a very ample linear system. Let E be a stable reflexive sheaf on X . Is $E|_D$ stable for general $D \in |mH|$ for $m \geq 4$?

Proof of the Theorems.

While the claim is about the stability of E when restricted to a general, hence smooth, curve in $|mH|$, we will prove stability for certain reducible curves.

Definition 5. Let X be a normal, projective surface, $|H|$ an ample and base point free linear system (not necessarily complete).

By a *nodal m -gon with sides in H* we mean a curve $C \subset X$ that is the union of m smooth members of $|H|$ and whose singularities are ordinary nodes.

We can view the space of all nodal m -gons either as a locally closed subvariety of $|mH|$ or as an open subvariety of $|H|^m$. The latter shows that it is an irreducible variety.

Proposition 6. *Let X be a normal, projective surface, $|H|$ an ample and base point free linear system (not necessarily complete). Fix a smooth point $x_0 \in X$ and a smooth curve $x_0 \in C_0 \in |H|$ such that X is smooth along C_0 . Let E be a stable reflexive sheaf of rank r on X such that $E|_{C_0}$ is locally free and semistable.*

Then the restriction of E to the general nodal m -gon with sides in H is stable for $m \geq \frac{1}{2}r^2 + 4$.

Proof. Being locally free and semistable are open properties, hence $E|_{C_\lambda}$ is locally free and semistable for general $C_\lambda \in |H|$.

Let $\mathcal{T} = \{T_{\lambda,\mu}\} \subset |3H|$ be the space of all nodal triangles $C_0 + C_\lambda + C_\mu$ where $C_\lambda, C_\mu \in |H|$, X is smooth along C_λ, C_μ and the restrictions $E|_{C_\lambda}, E|_{C_\mu}$ are locally free and semistable. Let $C_{\mathcal{T}} \subset X \times \mathcal{T}$ be the universal curve and $E_{\mathcal{T}} \rightarrow C_{\mathcal{T}}$ the pull-back of E to $C_{\mathcal{T}}$. We can also think of $E_{\mathcal{T}}$ as the universal vector bundle whose restriction to $T_{\lambda,\mu}$ is $E_{\lambda,\mu} := E|_{T_{\lambda,\mu}}$.

Let $D_{\mathcal{T}} \subset \text{Quot}(E_{\mathcal{T}})$ denote the subscheme parametrizing torsion free quotients $q_{\lambda,\mu} : E_{\lambda,\mu} \twoheadrightarrow F$ with the same slope as $E_{\lambda,\mu}$. Its fiber over $T_{\lambda,\mu}$ is denoted by $D_{\lambda,\mu} \subset \text{Quot}(E_{\lambda,\mu})$.

Note that $E_{\lambda,\mu}$ is semistable by (9) and each $q_{\lambda,\mu} : E_{\lambda,\mu} \twoheadrightarrow F$ induces 3 quotients

$$E|_{C_0} \rightarrow F_0, \quad E|_{C_\lambda} \rightarrow F_\lambda \quad \text{and} \quad E|_{C_\mu} \rightarrow F_\mu.$$

Each of these has the same slope as $E|_{C_0}$ and at each node $p \in C_0 + C_\lambda + C_\mu$ the two branches give the same quotient $E_p \rightarrow F_p$. In particular, any such $q_{\lambda,\mu} : E_{\lambda,\mu} \twoheadrightarrow F$ is uniquely determined by

$$q_{\lambda,\mu} \otimes k(x_0) : E_{x_0} \rightarrow F_{x_0}.$$

Therefore we can think of $D_{\lambda,\mu}$ as a subscheme of $\text{Quot}(E_{x_0})$, which is a union of Grassmannians of quotients of the vector space E_{x_0} . Thus

$$\dim D_{\lambda,\mu} \leq \dim \text{Quot}(E_{x_0}) \leq \frac{1}{4}r^2.$$

Note that since each $E_{\lambda,\mu}$ is semistable, $D_{\mathcal{T}} \rightarrow \mathcal{T}$ is proper.

We apply (8) to $D_{\mathcal{T}} \subset \mathcal{T} \times \text{Quot}(E_{x_0})$ and first we show that the alternative (8.1) is impossible if E is stable.

If (8.1) holds then $D_{\mathcal{T}} \rightarrow \mathcal{T}$ has a constant section. Equivalently, there is a quotient

$$q_{\mathcal{T}} : E_{\mathcal{T}} \rightarrow F_{\mathcal{T}},$$

such that, for every $T_{\lambda,\mu} \in \mathcal{T}$, its restriction gives a quotient (which has the same slope as $E_{\lambda,\mu}$)

$$q_{\lambda,\mu} : E_{\lambda,\mu} \rightarrow F_{\lambda,\mu} \quad \text{such that} \quad q_{\lambda,\mu} \otimes k(x_0) = q_0.$$

We use these to construct a quotient sheaf $E \rightarrow F$ whose pull-back to $C_{\mathcal{T}}$ is $F_{\mathcal{T}}$. This will then contradict the stability of E .

To construct $E \rightarrow F$, pick a general point $x \in X$ and let $C_{\lambda}, C_{\mu} \in |H|$ be general smooth curves through x . Set $q_x := q_{\lambda,\mu} \otimes k(x)$.

Note that $E|_{C_0+C_{\lambda}}$ is semistable, hence $q_{\lambda,\mu}|_{C_0+C_{\lambda}}$ is uniquely determined by $E|_{C_0+C_{\lambda}}$ and by q_0 . Thus q_x does not depend on the choice of C_{μ} . Similarly, it also does not depend on the choice of C_{λ} . Thus, as the notation suggests, q_x is independent of the choices of C_{λ}, C_{μ} . Thus we get a well defined quotient $q : E \rightarrow F$ such that $F|_{C_{\lambda}} = F_{\lambda,\mu}|_{C_{\lambda}}$ for general $C_{\lambda} \in |H|$. This shows that E is not stable, a contradiction. Therefore the alternative (8.2) must hold.

Fix $m_0 \geq \frac{r^2}{4} + 1$ and pick general pairs $C_{\lambda_i}, C_{\mu_i} \in |H|$. We claim that the restriction of E to the $(2m_0 + 1)$ -gon $C_{\Sigma} := C_0 + \sum_{i=1}^m (C_{\lambda_i} + C_{\mu_i})$ is stable.

Assume to the contrary that there is a quotient $q_{\Sigma} : E|_{C_{\Sigma}} \rightarrow F_{\Sigma}$ which has the same slope as $E|_{C_{\Sigma}}$. Let $q_0 : E_{x_0} \rightarrow F_{x_0}$ be the induced quotient on the fiber over x_0 . We can restrict q_{Σ} to

$$q_i : E|_{C_0+C_{\lambda_i}+C_{\mu_i}} \rightarrow F_i \quad \text{for } i = 1, \dots, m.$$

Each q_i gives a point in D_{λ_i,μ_i} and $[q_0] \in \cap_{i=1}^m D_{\lambda_i,\mu_i}$. This contradicts (8.2), hence $E|_{C_{\Sigma}}$ is stable.

The smallest value of m_0 we can take is $\lceil \frac{1}{4}r^2 \rceil + 1$, which gives that E restricted to the general m -gon is stable for $m \geq 2\lceil \frac{1}{4}r^2 \rceil + 3$. This holds if $m \geq \frac{1}{2}r^2 + 4$. \square

7 (Proof of (2)). By (6), E restricted to the m -gon is stable. Since stability is an open condition, E restricted to the general member of $|mH|$ is also stable. \square

Lemma 8. *Let U be an irreducible variety, V any variety and $Z \subset U \times V$ a closed subscheme. Then*

- (1) *either $U \times \{v\} \subset Z$ for some $v \in V$,*
- (2) *or $\cap_{i=1}^m Z_{u_i} = \emptyset$ for $m > \dim V$ and general $u_1, \dots, u_m \in U$.*

Proof. Assume that (1) fails. By induction we show that $\dim \cap_{i=1}^r Z_{u_i} \leq \dim V - r$ for $r \leq \dim V + 1$ and general $u_1, \dots, u_r \in U$.

This is clear for $r = 0$. To go from r to $r + 1$, note that none of the irreducible components of $\cap_{i=1}^r Z_{u_i}$ is contained in every Z_{u_r} . Thus, a general $Z_{u_{r+1}}$ contains none of them, hence

$$\dim \cap_{i=1}^{r+1} Z_{u_i} < \dim \cap_{i=1}^r Z_{u_i}. \quad \square$$

Proposition 9. *Let E be a stable (resp. semistable) bundle on X and let $C_1, \dots, C_r \in |H|$ be smooth curves such that the $E|_{C_i}$ are stable (resp. semistable). Then $E|_{C_1+\dots+C_r}$ is also stable (resp. semistable).*

Proof. This follows essentially from [TiB95, Prop.1.2]. The only thing to observe is that, since the curves all lie on the surface X , the weights that [Ses82, Sec.7] associates to torsion-free sheaves on $C_1+\dots+C_r$ for the purposes of defining semistability are all equal. From this, one observes that the inequality [TiB95, 1.1] is satisfied in our situation. \square

Remark 10. It is easy to modify the above results to get stability not only for general $C \in |mH|$ but also for general $C \in |mH|$ passing through some preassigned points.

Let F be stable on X and fix $x_1, \dots, x_r \in X$ such that F is locally free at these points. Let $p : Y \rightarrow X$ be the blow-up of X along the points x_1, \dots, x_r with exceptional divisor D .

Set $E = p^*F$. Then, as in [Buc00, Prop.3.4], for some $n \gg 1$, E is stable on Y with respect to the polarization $H_n := np^*(H) - D$. Fix this n and set $H_Y := H_n$.

Now apply the above results on stable bundles to get m such for $E|_{C_0}$ is stable for sufficiently general $C_0 \in |mH_Y|$.

Consider $E|_{C_0+D}$. Since $E|_D$ is trivial, it is semistable with respect to H_Y and since $E|_{C_0}$ is stable, by [TiB95, Prop.1.2] it follows that $E|_{C_0+D}$ is actually stable. Being stable is an open condition, hence we see that $E|_{C_1}$ is stable for a general member $C_1 \in |mH_Y + D| = |mp^*H - (m-1)D|$. Proceeding the same way, we get eventually that $E|_{C_{m-1}}$ is stable for a general member $C_{m-1} \in |mH_Y + (m-1)D| = |mp^*H - D|$. Set $C := p(C_{m-1})$ and note that in fact $C \cong C_{m-1}$. Thus $V|_C$ is stable and C is a smooth member of $|mH|$ passing through the points x_1, \dots, x_r .

11 (Proof of Theorem 3). The arguments are quite similar to the ones used to show Theorem 2, hence we only outline them.

As usual (see, for instance, [BK08, §3]), by passing to a finite cover of X if necessary, we may assume that $\det E$ is trivial.

Pick a general point $x_0 \in X$ and a curve $x_0 \in C_0 \in |H|$ such that $E|_{C_0}$ is locally free and polystable. By assumption $E|_{C_\lambda}$ is locally free and polystable for general $C_\lambda \in |H|$. Since $\deg E|_{C_\lambda} = 0$, by [NS65], one can also obtain $E|_{C_\lambda}$ from a unitary representation of the fundamental group of C_λ . In particular, there is a well defined notion of parallel transport along any path in C_λ or in any m -gon $\cup_i C_{\lambda_i}$ if $E|_{C_{\lambda_i}}$ is locally free and polystable for every i .

Let $\text{Hol}_{x_0}(E) \subset GL(E_{x_0})$ denote the holonomy group [BK08] and $\text{Hol}_{x_0}^\circ(E) \subset \text{Hol}_{x_0}(E)$ its identity component. By [BK08, 40], $\pi_1(X, x_0) \rightarrow \text{Hol}_{x_0}(E)/\text{Hol}_{x_0}^\circ(E)$ is surjective. By the Lefschetz theorem, $\pi_1(C_0, x_0) \rightarrow \pi_1(X, x_0)$ is surjective, which implies that

$$\text{Hol}_{x_0}(E|_{C_0}) \rightarrow \text{Hol}_{x_0}(E)/\text{Hol}_{x_0}^\circ(E) \quad \text{is surjective.} \quad (11.1)$$

Therefore, using [BK08, 40], by passing to a suitable finite étale cover of X we may assume that $\text{Hol}_{x_0}(E)$ is connected.

Let $\mathcal{T} = \{T_{\lambda,\mu}\} \subset |3H|$ be the space of all nodal triangles $C_0 + C_\lambda + C_\mu$ where $C_\lambda, C_\mu \in |H|$, X is smooth along C_λ, C_μ and the restrictions $E|_{C_\lambda}, E|_{C_\mu}$ are locally free and polystable.

Using the parallel transport along the 3 components, we get a (possibly non-unitary) representation

$$\rho_{\lambda,\mu} : \pi_1(C_{\lambda,\mu}, x_0) \rightarrow GL(E_{x_0}).$$

The images of all these representations generate a subgroup $H_{\mathcal{T}}$ of the holonomy group $\text{Hol}_{x_0}(E)$.

First we claim that $H_{\mathcal{T}} = \text{Hol}_{x_0}(E)$. Assume to the contrary that $H_{\mathcal{T}} \subsetneq \text{Hol}_{x_0}(E)$. Then there is a tensor power $E_{x_0}^{\otimes m}$ and a vector $w \in E_{x_0}^{\otimes m}$ that is $H_{\mathcal{T}}$ -invariant but not $\text{Hol}_{x_0}(E)$ -invariant. Thus, for each triangle $C_{\lambda,\mu}$ we get a flat section

$$w_{\lambda,\mu} \in H^0(C_{\lambda,\mu}, (E_{\lambda,\mu})^{\otimes m}).$$

As in the proof of Theorem 2 we see that for every $x \in C_{\lambda} \cap C_{\mu}$, the fiber $w_{\lambda,\mu}(x)$ depends only on x but not on C_{λ} and C_{μ} . Thus we get a well defined global section

$$w_X \in H^0(X \setminus (\text{finite set}), E^{\otimes m})$$

which then extends to a global section of (the reflexive hull of) $E^{\otimes m}$. Thus $w = w_X(x)$ is $\text{Hol}_{x_0}(E)$ -invariant, a contradiction. This proves that $H_{\mathcal{T}} = \text{Hol}_{x_0}(E)$.

Continuing with the method of Theorem 2 would give a bound that depends on r and on m above. In many important cases, for instance when $\text{Hol}_{x_0}(E) = SL(E_{x_0})$, one can choose $m = 2$ [BK08, Prop.5]. However, even this would give a degree 4 bound in r . In general, it is not known how to bound m effectively.

Thus, instead of trying to control the quot-scheme as in the proof of Theorem 2, we control the size of the holonomy group on m -gons using (12).

Choose $r^2 + 1$ general pairs (λ_i, μ_i) . Then, by (12), the images of ρ_{λ_i, μ_i} for $i = 1, \dots, r^2 + 1$ generate $\text{Hol}_{x_0}(E)$.

This implies that the restriction of E to the general $(2r^2 + 3)$ -gon is stable and has holonomy group $\text{Hol}_{x_0}(E)$. By the lower semicontinuity of the holonomy groups [BK08, §1], the same holds for a general smooth curve in $|(2r^2 + 3)H|$. \square

Lemma 12. *Let $G \subset GL(n, \mathbb{C})$ be a connected algebraic group of dimension d over \mathbb{C} and $S \subset G$ a connected (in the Euclidean topology) subset that generates a Zariski dense subgroup of G . Then there are $d + 1$ elements $s_0, \dots, s_d \in S$ that generate a Zariski dense subgroup of G .*

Proof. For $0 \leq r \leq d$ we use induction to find $s_0, \dots, s_r \in S$ such that the Zariski closure of $\langle s_0, \dots, s_r \rangle$ has dimension at least r . This is clear for $r = 0$.

To start with, fix any $s_0 \in S$ and consider ss_0^{-1} as a function $S \rightarrow G$. It sends s_0 to the identity. If the eigenvalues of ss_0^{-1} are not constant near s_0 , then for very general $s_1 \in S$, at least one of the eigenvalues of $s_1 s_0^{-1}$ is not a root of unity. Then $s_1 s_0^{-1}$ has infinite order, hence the Zariski closure of $\langle s_1 s_0^{-1} \rangle$ has positive dimension. If the eigenvalues of ss_0^{-1} are constant near s_0 , then all the eigenvalues of $s_1 s_0^{-1}$ equal 1. Thus $s_1 s_0^{-1}$ has infinite order, unless $s_1 = s_0$.

Now to the inductive step. Let $H_r \subset G$ denote the Zariski closure of $\langle s_0, \dots, s_r \rangle$ and $H_r^\circ \subset H_r$ its identity component. By assumption, $\dim H_r^\circ \geq r$.

If H_r° is a normal subgroup of G , then the above argument applies to G/H_r° . Pick $s_{0r} \in S \cap H_r^\circ$ such that no open neighborhood of $s_{0r} \in U \subset S$ is contained in H_r° . We obtain that the Zariski closure of $\langle s_{r+1} s_{0r}^{-1} \rangle$ is a positive dimensional subgroup of G/H_r° . Thus the Zariski closure of $\langle s_0, \dots, s_r, s_{r+1} s_{0r}^{-1} \rangle$ has dimension

at least $r + 1$. Since s_{0r} is in the Zariski closure of $\langle s_0, \dots, s_r \rangle$, we can replace $s_{r+1}s_{0r}^{-1}$ by s_{r+1} without changing the Zariski closure.

If H_r° is not a normal subgroup of G , then pick an s_{r+1} that is not contained in the normalizer of H_r° . Then H_r° is not a normal subgroup of the Zariski closure of $\langle s_0, \dots, s_{r+1} \rangle$. The identity component is always a normal subgroup, thus the identity component of the Zariski closure of $\langle s_0, \dots, s_{r+1} \rangle$ is strictly larger than H_r° . \square

Remark 13. (1) The connectedness of S is essential in (12). For instance, all the roots of unity generate a Zariski dense subgroup of \mathbb{C}^* , but any finite subset of them generates a finite subgroup.

(2) It is easy to see that 2 very general elements of a connected, reductive, algebraic group generate a Zariski dense subgroup. Indeed, the Zariski closure of the subgroup generated by a very general semisimple element g_1 is a maximal torus. The maximal torus acts on the Lie algebra of G with 1-dimensional eigenspaces (except on the Lie algebra of the torus), hence only finitely many connected subgroups contain any given maximal torus. Pick any $g_2 \in G$ not in the normalizer of any of these subgroups that are not normal in G . Then $\langle g_1, g_2 \rangle$ is a Zariski dense subgroup of G .

(3) Probably a small case analysis would improve the bound $\dim G + 1$ in (12) to $\dim G$ which is the optimal result for $G = \mathbb{C}^d$, where $d-1$ elements always generate a smaller dimensional subgroup. A very general element of $(\mathbb{C}^*)^d$ generates a Zariski dense subgroup, but if we take $S \subset (\mathbb{C}^*)^d$ to be the union of the “coordinate axes” $(1, \dots, 1, *, 1, \dots, 1)$ then again no $(d-1)$ -element subset of S generates $(\mathbb{C}^*)^d$.

We believe, however, that one can do much better for reductive groups, especially if $S \subset G$ is an irreducible real algebraic subset. Here the worst example we know is the following.

(4) The set of all reflections generate the orthogonal group $O(d)$ but $d-1$ reflections always have a common fixed vector, hence they generate a smaller dimensional subgroup. (The orthogonal group is not connected, so it may be better to work with the orthogonal similitudes and with scalars times reflections.)

Question 14. Let G be a connected, reductive algebraic group of rank r over \mathbb{C} and $S \subset G$ an irreducible, real, semialgebraic subset that generates a Zariski dense subgroup of G . Is it true that $2r$ very general elements $s_1, \dots, s_{2r} \in S$ generate a Zariski dense subgroup of G .

Remarks on Question 4.

More generally, one can investigate the following.

Question 15. Let X be a smooth, projective surface and $|H|$ an ample and base point free linear system. Under what conditions on $(X, |H|)$ can one guarantee that for every stable vector bundle E on X , the restriction $E|_C$ is stable for general $C \in |H|$?

We know very few examples of pairs $(X, |H|)$ where stability of restrictions fails. One such case is when a general $C \in |H|$ is rational or elliptic. This holds, among others, for $(\mathbb{P}^2, |\mathcal{O}_{\mathbb{P}^2}(1)|)$, $(\mathbb{P}^2, |\mathcal{O}_{\mathbb{P}^2}(2)|)$ and $(\mathbb{P}^2, |\mathcal{O}_{\mathbb{P}^2}(3)|)$.

On a rational curve every stable bundle has rank 1 and on an elliptic curve every stable bundle with $c_1(E) = 0$ has rank 1. Thus if $\text{rank } E \geq 2$ and $c_1(E) = 0$ then $E|_C$ is never stable.

One can get more complicated examples out of these. Take any surface X and a general finite morphism $\pi : X \rightarrow \mathbb{P}^2$. Set $|H| := \pi^*|\mathcal{O}_{\mathbb{P}^2}(3)|$. Note that $|H|$ is ample but usually neither very ample nor complete. There are, however, many examples, for instance double covers whose branch locus has degree ≥ 8 , where the pulled-back $|H|$ is a complete linear system whose general member is a smooth curve of high genus. Nonetheless, if E is the pull-back of a vector bundle from \mathbb{P}^2 , then the restrictions $E|_C$ are not stable for $C \in |H|$.

These examples all satisfy $\dim |H| \leq 9$, but pulling back $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, a)|$ gives similar examples where both $\dim |H|$ and the genus of the general $C \in |H|$ are arbitrarily high.

These types are the obvious examples where general restrictions are not stable. We do not know any other.

Let us next turn to a heuristic argument that suggested Question 4 to us. We focus on the holonomy groups and propose the following variant.

Question 16. Let X be a smooth projective surface and $|H|$ a very ample linear system. Let E be a stable vector bundle on X . Is $E|_D$ stable for general $D \in |mH|$ for $m \geq 4$ and with the same holonomy group as E ?

As we saw in (11.1), the discrete part of the holonomy $\text{Hol}(E)/\text{Hol}^\circ(E)$ never causes problems in (16). Thus, by [BK08, 40], we can focus on the case when $\text{Hol}(E)$ is connected and $\det E \cong \mathcal{O}_X$.

For $m \gg 1$ take a general $x \in C \in |mH|$ such that $E|_C$ is stable and with the same holonomy group as E . Thus we get a holonomy representation

$$\rho_C : \pi_1(C) \rightarrow \text{Hol}(E)$$

whose image is Zariski dense.

Although not supported by any evidence, one can hope that in our situation C can be written as a connected sum $C = C_2 \# C'$ where the genus of C_2 is 2 and

$$\rho_2 : \pi_1(C_2) \rightarrow \text{Hol}(E) \quad \text{still has Zariski dense image.}$$

It is then another entirely uncorroborated belief that this C_2 can be realized by vanishing cycles as the curve acquires an ordinary 4-fold point. Eventually, this may lead to an approximation of $\rho_2 : \pi_1(C_2) \rightarrow \text{Hol}(E)$ by some $\rho_t : \pi_1(C_t) \rightarrow \text{Hol}(E)$ where $C_t \in |4H|$ is a family of curves whose limit also has an ordinary 4-fold point.

We stress that for the moment all this is just wishful thinking. We, however, feel that this approach raises many interesting questions that – even if Questions 4 and 16 turn out to be utterly misguided – could lead to a much improved understanding of stable bundles and their restrictions.

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